

Chapter 4

Section 4.2 – Some Existence Theorems

Find the coordinates of all relative extrema:

$$x^2 - xy + y^2 = 3$$

$$2x - y - x \frac{dy}{dx} + y(-1) + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

$$\textcircled{1} \quad \frac{dy}{dx} = 0$$

$$\begin{aligned} y - 2x &= 0 \\ \therefore y &= 2x \end{aligned}$$

$$\begin{aligned} x^2 - x(2x) + (2x)^2 &= 3 \\ x^2 - 2x^2 + 4x^2 &\stackrel{?}{=} 3 \\ 3x^2 &= 3 \\ x &= \pm 1 \end{aligned}$$

$$(1, 2) \quad (-1, 2)$$

$$\textcircled{2} \quad \frac{dy}{dx} \Rightarrow \phi$$

$$\begin{aligned} 2y - x &= 0 \\ 2y &= x \end{aligned}$$

$$\begin{aligned} (2y)^2 - (2y)y + y^2 &= 3 \\ 3y^2 &= 3 \\ y &= \pm 1 \end{aligned}$$

$$(2, 1) \quad (-2, 1)$$



Find the coordinates of all relative extrema (cont.):

$$x^2 - xy + y^2 = 3$$

$$\frac{dy}{dx} = \frac{y-2x}{2y-x}$$

$$\frac{d^2y}{dx^2} = \frac{(2y-x)\left(\frac{dy}{dx}-2\right) - (y-2x)\left(2\frac{dy}{dx}-1\right)}{(2y-x)^2}$$

$$\frac{d^2y}{dx^2}\Big|_{(1,2)} = \frac{(2\cdot 2-1)(0-2) - (2-2)(2\cdot 0-1)}{(2\cdot 2-1)^2} = \frac{-6}{9} < 0$$

∴ REL max @ (1,2)

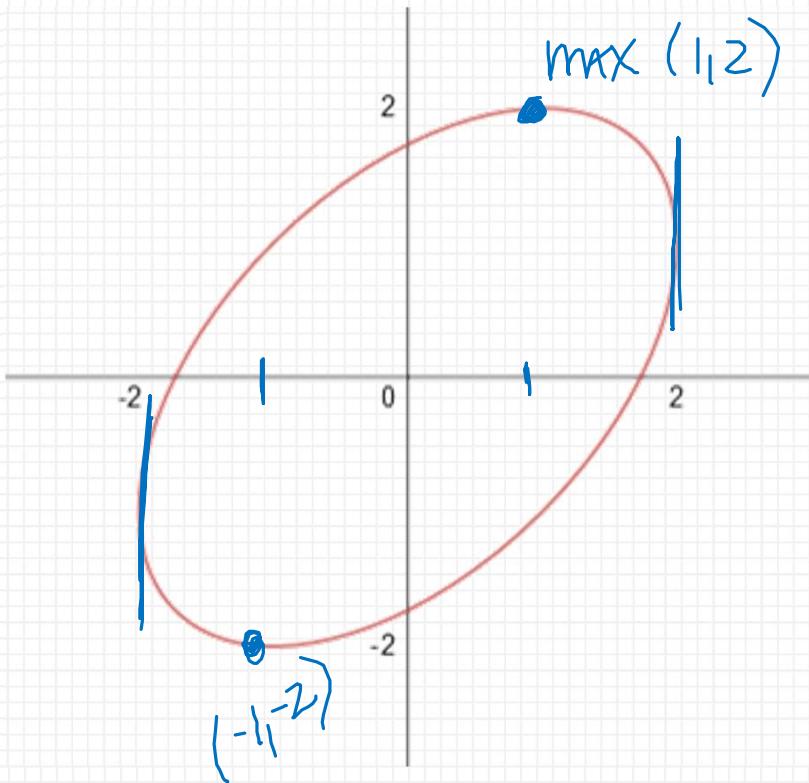
$$\frac{d^2y}{dx^2}\Big|_{(-1,-2)} = \frac{(-4+1)(0-2) - (-2+2)(2\cdot 0-1)}{(-4+1)^2} = \frac{6}{9} > 0$$

∴ REL-min @ (-1,-2)

$$\frac{d^2y}{dx^2}\Big|_{(z_1)} \text{ DNE } \text{ SINCE } \frac{dy}{dx}\Big|_{(z_1)} \text{ DNE,}$$



Graphical Support:



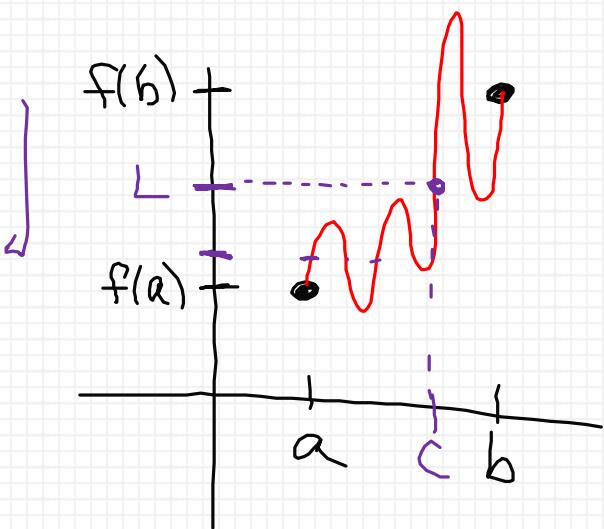
Absolute (Global) Extrema

"FOR ALL"

- ① IF $f(c) \geq f(x)$ $\forall x \in$ DOMAIN OF $f \Rightarrow f(c)$ IS MAX VALUE OF $f(x)$.
- \downarrow
- "AN ELEMENT OF"
- ABSOLUTE max occurs @ $x = c$.
- ② IF $f(c) \leq f(x)$ $\forall x \in$ DOMAIN OF $f \Rightarrow f(c)$ IS MIN VALUE OF $f(x)$
- ABSOLUTE min occurs @ $x = c$.



Intermediate Value Theorem (IVT)

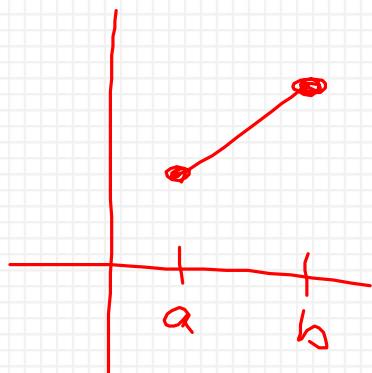


- ① IF f IS CNT ON $[a,b]$ THEN $f(x)$ MUST TAKE ON ALL VALUES BETWEEN $f(a)$ AND $f(b)$
- ② IF f IS CNT. ON $[a,b]$ $\Rightarrow \forall L \in [f(a), f(b)]$
 $\exists c \in (a,b)$ SUCH THAT $f(c) = L$

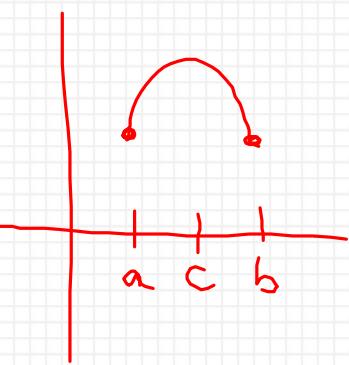


Extreme Value Theorem (EVT)

If $f(x)$ is cont on $\underline{[a,b]}$ \Rightarrow $f(x)$ has a maximum value and a minimum value for $x \in \boxed{[a,b]}$



$$\begin{aligned} \text{MAX VALUE} &= f(b) \\ \text{MIN VALUE} &= f(a) \end{aligned}$$



$$\begin{aligned} \text{MAX VALUE} &= f(c) \\ \text{MIN VALUE} &= f(a) = f(b) \end{aligned}$$

max/min values occur @

① endpoint, or

② crit. pt



Let g be a continuous function on the closed interval $[-2, 4]$. A few values of g are given in this table:

x	-2	0	2	4
$g(x)$	1	-2	3	-5

IVT

Which intervals *must* contain a solution to $g(x) = -1$?

① $[-2, 0]$

② $[0, 2]$

③ $[2, 4]$



Find the maximum and minimum values of the function on the given interval. (EVT)

$$f(x) = -x^3 + 12x + 5, [-3, 3]$$

POLY. PXN CMT M $[-3, 3]$

$$f' = -3x^2 + 12 = 0$$

$$3x^2 = 12$$

$$x^2 = 4$$

$$x = \pm 2$$

CANDIDATES TEST

$$f(-3) = -4$$

MAX VALUE IS 21

$$f(-2) = -11$$

@ $x = 2$

$$f(2) = 21$$

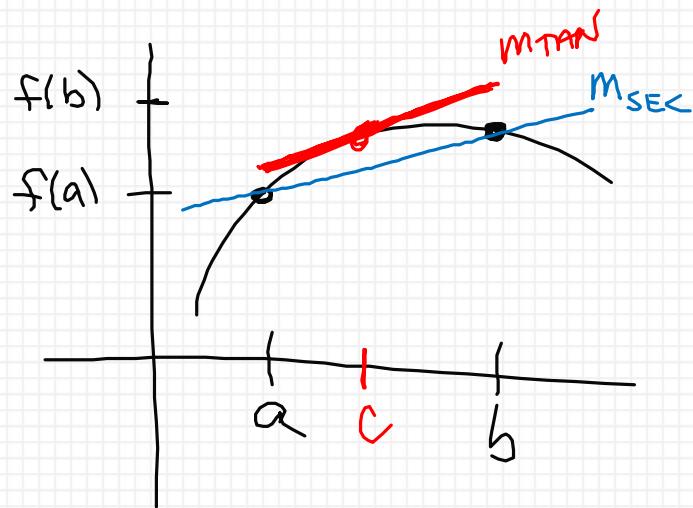
MIN VALUE IS -11

$$f(3) = 14$$

@ $x = -2$



Mean Value Theorem (MVT)

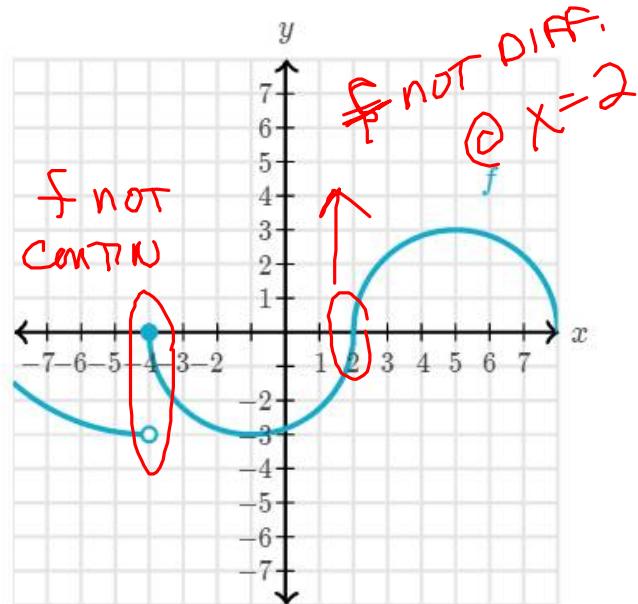


IF f IS CMT ON $[a,b]$ AND
DIFFERENTIABLE ON $(a,b) \Rightarrow$
 $\exists c \in (a,b)$ S.T. $m_{\text{TAN}}|_{x=c} = m_{\text{SEC}}[a,b]$

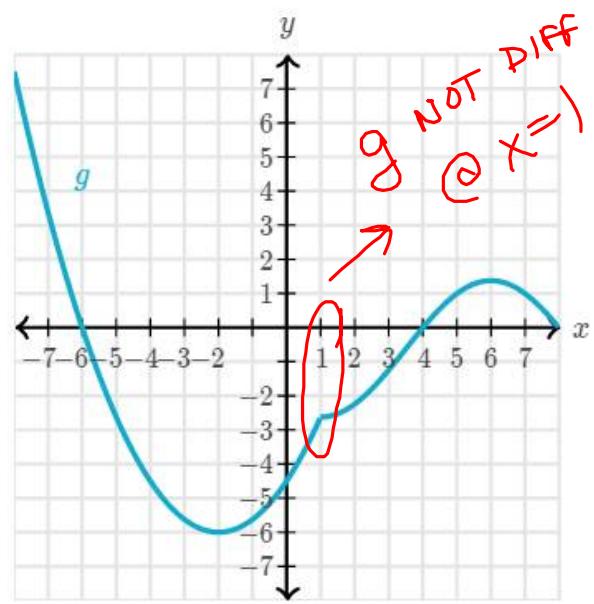
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



The graph of function f has a vertical tangent at $x = 2$.



The graph of function g has a sharp turn at $x = 1$.



x	-2	-1	0	1
$g(x)$	10	7	8	3

Rafael said that since $\frac{g(1) - g(0)}{1 - 0} = -5$ there must be a number c in the interval $[0, 1]$ for which $g'(c) = -5$.

Which condition makes Rafael's claim true?

① g CMT on $[0, 1]$

② g DIFF. on $(0, 1)$

Let $g(x) = x^3 - 16x$ and let c be the number that satisfies the Mean Value Theorem for g on the interval $[-4, 2]$.

What is c ?

$g(x)$ is a polynomial $\Rightarrow g(x)$ is cont. and differentiable $(-\infty, \infty) \Rightarrow$ MVT APPLIES

$$g' = 3x^2 - 16 = \frac{g(2) - g(-4)}{2 - (-4)}$$

$$3x^2 - 16 = -\frac{24 - 0}{6}$$

$$3x^2 - 16 = -4$$

$$3x^2 = 12$$

$$x^2 = 4$$

$$x = \pm 2$$

$$c = -2$$

*~~x=2~~ is an endpoint and
 $c \in (-4, 2)$

Find the value that satisfies the MVT on the given interval.

$$f(x) = x^2 + x, \quad [-4, 6]$$

$$f(x) = \frac{1}{x-1}, \quad [0, 3] \quad f = (x-1)^{-1}$$

$$\frac{f(3) - f(0)}{3 - 0} = \frac{1/2 + 1}{3} = \frac{1}{2}$$

$f' = -(x-1)^{-2} = -\frac{1}{(x-1)^2} = -\frac{1}{2}$

$-(x-1)^2 = 2$

MVT DOES NOT APPLY
f NOT CMA.
@ $x=1$



Homework/Classwork:

AP Packet #1-4, 14, 25-29 odd, 37-40

